International Journal of Technology (2017) 3: 437-447
ISSN 2086-9614

A NONLINEAR ANALYTICAL MODEL FOR SYMMETRIC LAMINATED BEAMS IN THREE-POINT BENDING

Youssef Benbouras[1*], Aziz Maziri[1], Elhassan Mallil[1], Jamal Echaabi[1,2]

[1]*Équipe de Recherche Appliquée sur les Polymères, Département de Génie Mécanique, ENSEM, Université Hassan II Aïn Chok, BP 8118, Oasis, Casablanca, Maroc*
[2]*Université Mohammed 6 des Sciences de la Santé, Ecole Supérieure de Génie Biomédical Avenue Ali Ibnou Abi Taleb, Quartier de la ligue Arabe, 20000, Casablanca*

(Received: September 2016 / Revised: January 2017 / Accepted: March 2017)

ABSTRACT

The use of composite materials with continuous fibers in the aircraft and aerospace industries requires a thorough knowledge of behaviors of these laminate composites under various loading conditions. Indeed, the aim of this work is to simulate linear and nonlinear behavior of a symmetric laminated composite under three-point bending tests. The modelization used is based on an analytical approximation that has been recently developed for isotropic materials. This approximation is still valid for the studied quasi-isotropic laminated composite because it is symmetric with a specific layer sequence. The overall response of laminate composite is determined from the behavior of each ply outside of their orthotropic axis. Two methods are used to calculate the equivalent longitudinal Young-modulus of the laminate. The result shows that when the deflection of the specimens is less than 2.5 times the thickness, the difference between the experimental and analytical curves is about 1% for the average global stresses method, and about 7.5% for the apparent bending modulus method. For large deflections, the difference relative to the first method remains less than 11% and the second method is about 20%.

Keywords: Analytical model; Graphite-epoxy composite; Large deflection; Linear and nonlinear behavior; Three-point bending test

1. INTRODUCTION

Laminated composites are now widely used in advanced fields where greater accuracy requirements for modeling the mechanical behavior and design are increasingly required. Accurate knowledge of mechanical properties of a laminate structure is therefore fundamental. Indeed, the reinforcement-matrix mixture represents the basic structure of laminates, and there are several relationships for calculating the elastic constants of the laminate structure. Such characteristics are very sensitive to conditions for carrying out the testing of composite parts. It is therefore necessary to have data provided by suppliers of reinforcements and matrices, or to have obtained on par experimental tests conducted in the laboratory. The biaxial test is the ideal to validate the macroscopic behavior of composite structures. However, this type of testing is used less often because it is difficult to achieve and very expensive. An alternative way to biaxial testing is the three-point bending test that allows, in the case of some configurations, tracking the damage progression of a given specimen to the final failure point. Many authors, such as Vargas and Mujika (2011), Xiwen et al. (2013) and Moreno et al. (2016) were

*Corresponding author's email: youssef.mecanique@gmail.com Tel: +212657561110
Permalink/DOI: https://doi.org/10.14716/ijtech.v8i3.5786

interested in this kind of testing. In our case study, we use the experimental results developed by Echaabi et al. (1996). The latter observed various behaviors by varying the distance between the supports and the geometrical dimensions of the specimens in three-point bending tests. The non-linearity depends on the thickness ratio *l/h* effect, the orthotropy (having elastic properties in two or three planes, perpendicular to each other), the boundary conditions and the number of layers in the laminate. Our study is limited only to the thickness ratio *l/h*. A small ratio *l/h* leads to a linear behavior, whereas a larger ratio *l/h* presents a non-linear response (Dash & Singh, 2010).

Irhirane (Irhirane, 2007; Irhirane et al., 2010) used two formulations: an analytical method with transverse shearing and a finite element method. The first-order shear deformation theory (FSDT), which takes into account the transverse shearing strain with the implementation of corrective co-efficients, remains the best approach to characterize and simulate analytically the macroscopic curves and the sequences of failure for the two specimens 'A' and 'D' tested (Echaabi et al., 1996). Moreover, the use of the finite element method improves significantly the results on the level of breaking loads and flexural stiffness (Irhirane, 2007). The results of Irhirane's work permit obtaining a good correlation with experimental curves in the linear behavior. The present work is focused on modeling the behavior of specimens with a non-linear response which has not been studied as yet.

The work of Werren and Norris (1959) concerned a quasi-isotropic composite laminate composed of glass fabric and polyester resin in which the sequence contained directions of 0°, ±60°, 120°. Moreover, for membrane isotropy, the number of layers at 60° must be equal to that of 60° and the same in each direction. These laminates are widely used, especially in aerospace because, by definition of isotropic, they offer good uniformity in stiffness and strength in all directions, and also a good opposition to the propagation of cracks (Vannucci & Verchery, 2001a). Furthermore, Werren and Norris (1959) gave the first, simple rule, sufficient, but not necessary for the isotropic laminates, if layers are to have uni-directional reinforcement, the laminate must have the same number q of layers in m different orientations, with $m \geq 3$, offset by a constant angle equal to π/m. Also, Vannucci and Verchery (2001b) gave other additional conditions to construct isotropic laminates by a polar method. Some exact solutions were given by Vong and Verchery (1986) with a piling sequence of 48 layers.

Our work consists of the following steps: first, to calculate the equivalent longitudinal elastic modulus of an isotropic laminate by the global average stresses method and the apparent flexural modulus method (Gay, 1997). Thereafter, this modulus is substituted in equations describing approximately the center-deflection of the homogeneous isotropic specimens under large deflection. This approximate approach, developed by Venetis and Sideridis (2015), describes correctly the center-deflection of three-point bending specimens using formalism reduced to two equations. It also has the advantage of being easy in application and perfectly suited to the usual engineering practices.

2. METHODOLOGY

2.1. Basic Formulations and Behavior Law

A typical graphite/epoxy composite beam with n layers is shown in Figure 1 below.

In the literature, there are many theories for modeling the non-linear behavior of laminated composite. These theories can be divided into three main categories (Thai & Kim, 2015): the Equivalent Single Layer (ESL), Layer-Wise (LW) and Zig-Zag (Fares & Elmarghany, 2008). Three other theories can be deduced from ESL: the classical plate theory (CPT), the first-order shear deformation theory (FSDT), which takes into account the transverse shearing strain (TSS) and higher-order shear deformation theories (HSDT). Other simplified theories have been recently developed with fewer variables (Thai & Choi, 2013).

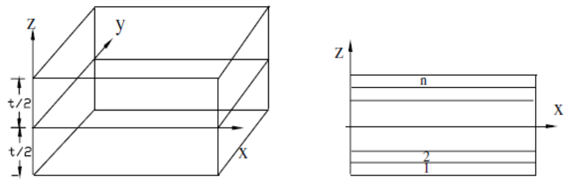

Figure 1 A graphite-epoxy beam with n layers

The previous theories are usually sufficient to analyze the elastic behavior of industrial structures. In our case, we used a modeling based on a first degree scheme which takes into account the transverse shearing strain and the correction of coefficients (Thai & Kim, 2015; Aydogdu, 2009). The displacement is given by Equation 1.

$$\begin{cases} u(x,y,z) = u^0(x,y) + z\,\theta_x(x,y) \\ v(x,y,z) = v^0(x,y) + z\,\theta_y(x,y) \\ w(x,y,z) = w^0(x,y) \end{cases} \tag{1}$$

where u, v and w are displacements of the specimen in directions x, y and z respectively. u^0, v^0 and w^0 are displacements of the mid-plane and θ_x , θ_y are rotations around axes x and y.
The strain field is given by the following Equation 2.

$$\varepsilon^T = \left\{ \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right\} \tag{2}$$

Substituting Equation 1 into Equation 2, the strain field can be expressed as shown in Equation 3.

$$\varepsilon = \varepsilon_m + z\varepsilon_b \tag{3}$$

where the membrane strain and the bending strain fields are defined by Equation 4.

$$\varepsilon_m^T = \left\{ \frac{\partial u^0}{\partial x}, \frac{\partial v^0}{\partial y}, \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} \right\} \text{ and } \varepsilon_b^T = \left\{ \frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right\} \tag{4}$$

The transverse shear is given by Equation 5.

$$\gamma^T = \{\gamma_{xz}, \gamma_{yz}\}^T = \left\{ \theta_x + \frac{\partial w}{\partial x}, \theta_y + \frac{\partial w}{\partial y} \right\} \tag{5}$$

The macroscopic behavior law of laminate is given by the following Equation 6 (Irhirane, 2007; Irhirane et al., 2010).

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{44} & F_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{6}$$

A_{ij} (i, j = 1, 2, 6) : the membrane stiffness matrix,
D_{ij} (i, j = 1, 2, 6) : the bending stiffness matrix,
B_{ij} (i, j = 1, 2, 6) : the membrane/bending coupling stiffness matrix,
F_{ij} (i, j = 4,5) : the transverse shear stiffness matrix,
M_x , M_y et M_{xy} : the bending moments,
N_x , N_y et N_{xy} : the membrane forces,
Q_x et Q_y : the transverse shear forces

2.2. Experimental Procedure

In this study, the material used is a graphite-epoxy laminate with a layer sequence of [[+45/-45/90/0]3]s (Figure 1). Dimensions of the test specimens and their mechanical properties are given respectively in Table 1 and Table 2 (Figure 6a).

Table 1 Geometrical characteristics of specimens in mm

Specimen	Length *l*	Distance l between supports	Width *b*	Thickness *h*	Thickness ratio *l*/*h*
A	75	57.5	25	3.6	16
B	150	115.0	25	3.6	32
C	150	136.5	25	3.6	38
D	75	57.5	10	3.6	16
E	150	115.0	10	3.6	32
F	150	136.5	10	3.6	38

Table 2 Mechanical properties of the laminate

Longitudinal elastic modulus E_{LL} (MPa)	116000
Transverse elastic modulus E_{TT} (MPa)	6900
Poisson's ratio ν_{LT}	0.3
Shear modulus G_{LT} (MPa)	5600

3. RESULTS AND DISCUSSION

3.1. Linear Behavior

The support span length-to-specimen thickness ratio *l*/*h* = 16 for Specimens A and D allows us to predict a linear behavior of the variation of the center-deflection w_c depending on the load P (Echaabi et al., 1996; Dash & Singh, 2010) (Figure 2). The Modeling is based on a first degree scheme which takes into account the transverse shearing strain and the correction of coefficients. The center deflection is given by the following relationship (Irhirane 2007; Irhirane et al., 2010) (Figure 2), which is shown in Equation 7.

$$w_c = \frac{Pl^3}{48b} D_{11}^* \left[1 + 12 \frac{F_{55}^*}{D_{11}^*} \frac{1}{l^2} \right] \quad (7)$$

w_c : the center deflection,
P : the applied load,
b : the width of the specimen,
l : the length of the specimen,
D_{11}^* : element of the inverse of the laminate flexural stiffness matrix,
F_{55}^* : element of the inverse of the transverse shear stiffness matrix

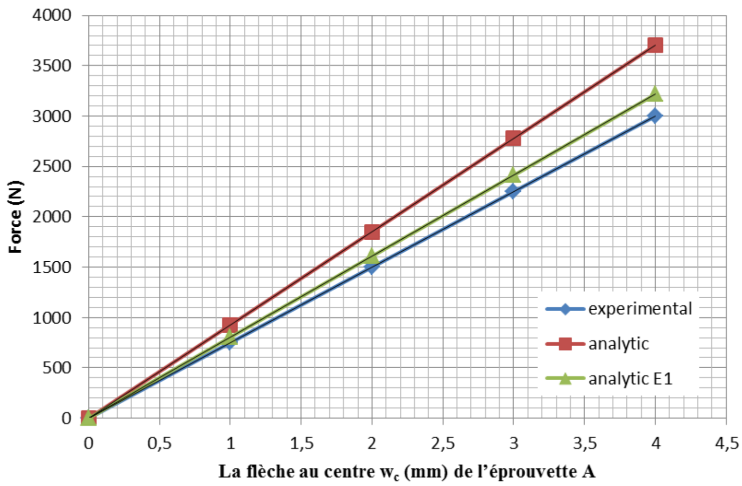

Figure 2 Variation of the center-deflection w_c according to the load P of the test specimen A

3.2. Nonlinear Behavior

The Specimens E and F, with a relatively large ratio $l/h = 32$ and $l/h = 38$, allow us to predict a non-linear behavior of the variation of the center-deflection w_c depending on the load P (Echaabi et al., 1996; Dash & Singh, 2010). The center-deflection of specimens is analytically determined by an approximate approach. This approach is based on two equations linking the intrinsic characteristics of homogeneous isotropic beams and the angle α (α is the angle of rotation of the mid-plane with respect to the y axis at supports). First, we calculated the equivalent longitudinal modulus of the laminate considered as isotropic by two different methods (Gay, 1997) as shown below.

3.2.1. Method of the average global stresses (E_1)

The law of membrane behavior is determined from the general law of the laminate as shown in Equation 8.

$$\begin{Bmatrix} N_x \\ N_y \\ T_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \varepsilon_{xy}^0 \end{Bmatrix} \quad (8)$$

It is possible to replace the force flows N_x, N_y and T_{xy} by average global stresses. Then we can deduce from Equation 8 the law of laminate membrane behavior made "homogeneous" as:

$$\begin{Bmatrix}\sigma_x^0\\ \sigma_y^0\\ \tau_{xy}^0\end{Bmatrix}=\begin{Bmatrix}N_x/h\\ N_y/h\\ T_{xy}/h\end{Bmatrix}=\frac{1}{h}\begin{bmatrix}A_{11} & A_{12} & A_{16}\\ A_{12} & A_{22} & A_{26}\\ A_{16} & A_{26} & A_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \varepsilon_{xy}^0\end{Bmatrix} \quad (9)$$

Terms of the above matrix $l/h[A]$ can be written as shown in Equation 9 and below as in Equation 10.

$$\frac{1}{h}A_{ij}=\sum_{k=1}^{n}E'^{k}_{ij}\frac{e_k}{h} \quad (10)$$

The ratios e_k/h can be re-arranged to form the proportions P^{θ} of layers having an orientation θ. In our case, the layer sequence used is [[+45/-45/90/0]3]s (Figure 3) and as shown in Equation 11.

Hence: $$E_1=\frac{1}{h}A_{11}=E'^{0°}_{11}P^{0°}+E'^{+45°}_{11}P^{+45°}+E'^{-45°}_{11}P^{-45°}+E'^{90°}_{11}P^{90°} \quad (11)$$

Knowing that the longitudinal elastic modulus E'^{θ}_{11} of a layer out of these axes of orthotropy is given by (Gay, 1997) in Equation 12, (See Figure 4).

$$\frac{1}{E'^{\theta}_{11}}=\frac{(\cos\theta)^4}{E_L}+\frac{(\sin\theta)^4}{E_T}+(\cos\theta\sin\theta)^2\left(\frac{1}{G_{LT}}-2\frac{\nu_{LT}}{E_T}\right) \quad (12)$$

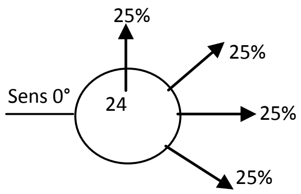

Figure 3 Laminate symbolization

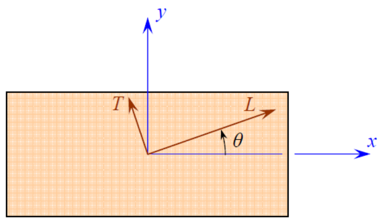

Figure 4 Main axes of laminate and orthotropic axes of a layer

3.2.2. The apparent bending modulus method (E_2)

The apparent bending modulus is obtained from comparison with the constitutive relations of a considered isotropic and homogeneous laminate. Let us consider two beams: the first is from homogeneous and isotropic material and the second is from graphite-epoxy laminate with *n* layers such as illustrated in Figures 5a and 5b. The deflection is given by the following Equations 13a, 13b, and 14.

$$\frac{d^2w_0}{dx^2}=-\frac{M_f}{EI}=-\frac{12M_x}{Eh^3} \quad (13a)$$

$$\frac{\partial^2 w_0}{\partial x^2}=-D'_{11}M_x \quad (13b)$$

Therefore : $$E_2=E_{fx}=\frac{12}{D'_{11}h^3} \quad (14)$$

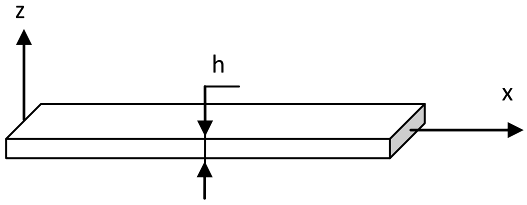

Figure 5a Homogeneous and isotropic beam

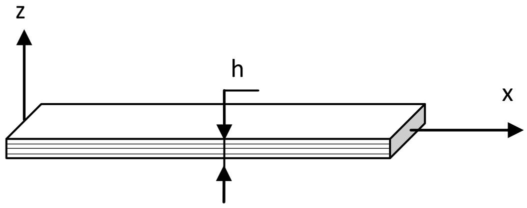

Figure 5b Graphite-epoxy laminated beam

3.2.3. The approximate approach describing the center-deflection of three-point bending test

The non-linearity observed for high values of the thickness ratio *l*/*h* is mainly generated by the lateral component of the reaction X' (Figure 6b). Indeed, this component depends on α which is the angle of rotation of the mid-plane with respect to the *y* axis at supports. This normal force of compression X' tends to emphasize the deflection by deforming the beam in a direction perpendicular to the compression axis. However, for a small deflection or a small ratio *l*/*h* the angle α is still low and the reactions can be considered perpendicular to the beam, this leads to a linear behavior. The work of Venetis and Sideridis (2015) consists of introducing this buckling effect in an analytical model. By studying the equilibrium of a beam subjected to the three-point bending test, as shown in (Figure 6a), these authors developed two main Equations, first based on Equation 15 giving the center-deflection w_c and the corresponding load *P* as a function of the angle α and two integrals that are solved approximately. Thus, these authors arrived at a final model reduced to two Equations 16 and 17, which are easy to apply as shown below.

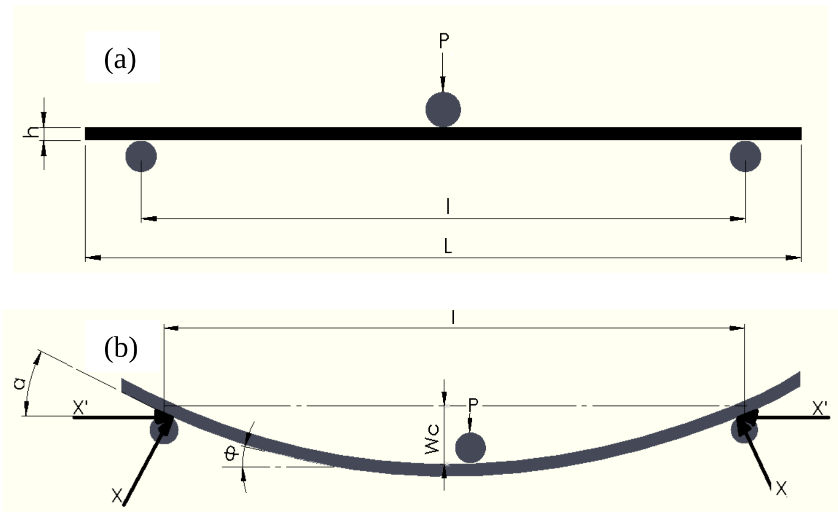

Figure 6 Experimental setup

$$\frac{1}{2}\left(\frac{Pl^2}{EI\cos\alpha}\right)^{\frac{1}{2}} = -2\cos\alpha\sqrt{\sin\alpha} + \frac{2}{3}\sin\alpha\sqrt{\sin\alpha}^{3} + \frac{1}{7}\sin\alpha\sqrt{\sin\alpha}^{7} + \frac{3}{44}\sin\alpha\sqrt{\sin\alpha}^{11} \tag{16}$$

$$\frac{w_c}{l}\left(\frac{Pl^2}{EI\cos\alpha}\right)^{\frac{1}{2}} = -2\sin\alpha\sqrt{\sin\alpha} + \frac{2}{3}\cos\alpha\sqrt{\sin\alpha}^{3} + \frac{1}{7}\cos\alpha\sqrt{\sin\alpha}^{7} + \frac{3}{44}\cos\alpha\sqrt{\sin\alpha}^{11} \tag{17}$$

3.3. Application and Analysis

In our case study, $D'_{11} = 8.65\ 10^{-6}(MPa.mm^3)^{-1}$, $E'^{0°}_{11} = 116000\,MPa$, $E'^{90°}_{11} = 6900\,MPa$, $E'^{\pm 45°}_{11} = 4078.91\,MPa$, $E_1 = 32764.45\,MPa$ and $E_2 = 29719.01\,MPa$.

After substituting the equivalent longitudinal modulus E1 or E2 in Equations 16 and 17 and resolving them, we deducted the value of the center-deflection w_c, the applied load P and the angle α. The result obtained shows a good correlation between the experimental and analytical curves for Specimens E and F. This is shown in Figure 7.

The modulus of elasticity E1, is calculated by assimilating each layer to a spring by neglecting the interlaminar cohesion (Equations 10 and 11), which explains the difference observed in a large deflection. Hence, the modulus of elasticity E2, is calculated approximately from the elementary rigidity relative to each layer (Equation 14). That approximation has generated a large difference between the theoretical and experimental curve.

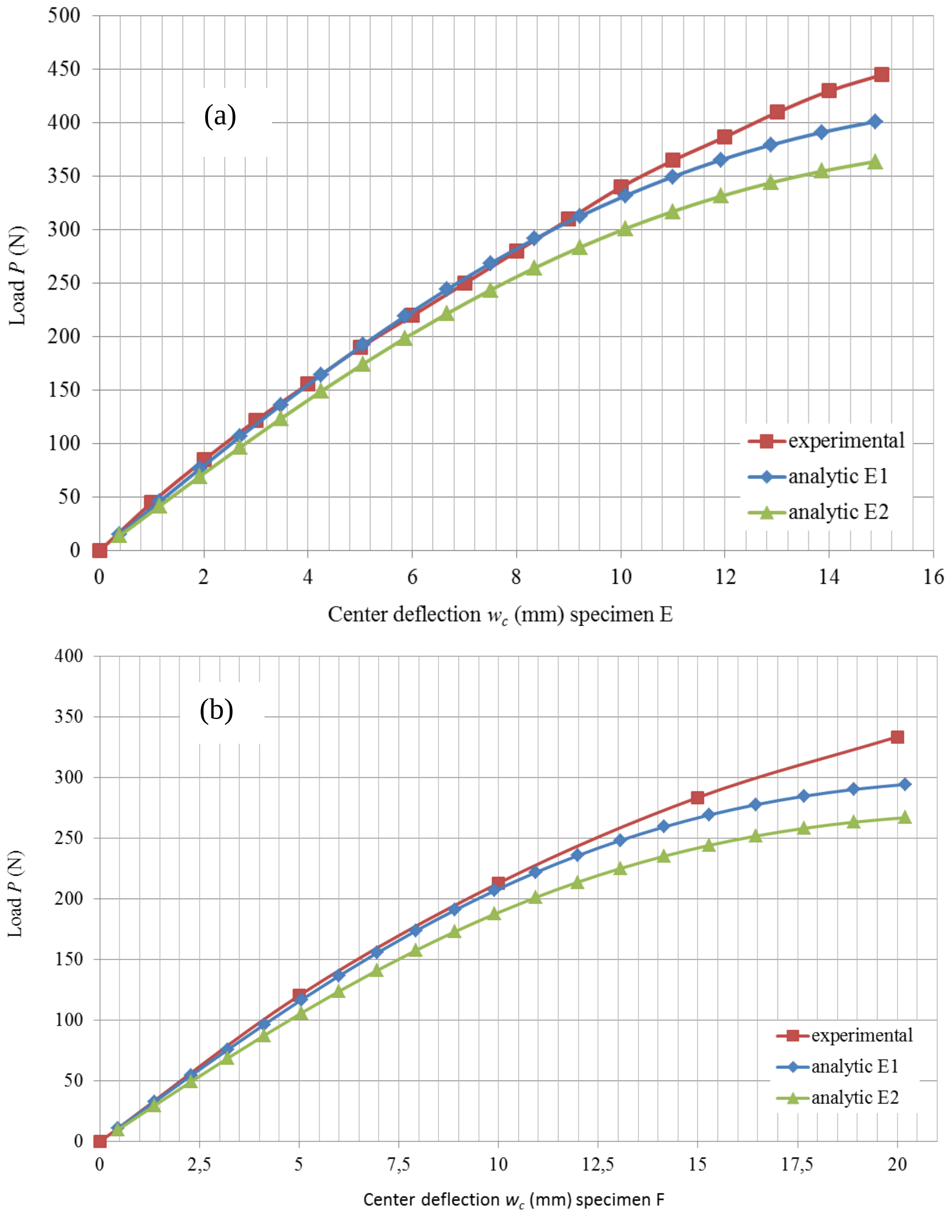

Figure 7 Variation of the center-deflection w_c according to the load P for: (a) Specimens E; and (b) Specimens F

Many authors such as Zhang and Kim (2005), Nguyen-Van et al. (2014), Zhang and Cheung (2003) and Tran et al., (2015) have developed numerical and analytical approaches to link the state of the non-linear behavior bending with the main characteristics of the specimens. These methods have a large number of unknown parameters to be determined which make it difficult to determine how to use it in the present study. We proposed an analytical model of the three-point bending test of a symmetrical laminated material using a simple approximate approach, which accurately describes the center-deflection of specimens in a large deflection. Macroscopic experimental curves show linear behavior for the small ratio *l*/*h* which was modeled by Irhirane (Irhirane, 2007; Irhirane et al., 2010). However, the non-linear behavior observed for some values of the thickness ratio *l*/*h* has not been modeled until the present time. In this work, we applied the theory of Werren and Norris (1959) in order to consider the laminate as isotropic. Then by two methods, namely the average global stresses and the apparent bending modulus, we deducted the equivalent; then we introduced the analytical approach developed by Venetis and Sideridis (2015) for homogeneous beams under wide deflection; (This approach is still valid for symmetrical laminates with a quasi-isotropic response). The obtained results, by substituting the equivalent longitudinal elastic modulus into equations of the approximate approach, predict the non-linear experimental curves with greater accuracy. However, this approach remains valid only in those laminates having the same number q of layers in m different orientations, with $m \geq 3$, offset by a constant angle equal to π/m.

The material undergoes a successive damage before the first macroscopic failure. Indeed, microscopic failures may be observed first in the matrix but not in the fiber. The follow-up to this work is to introduce failure criteria to predict the failure mode and the stress associated with the first macroscopic failure. This would allow studying the impact of material damage to the beam's structural behavior before the first failure.

4. CONCLUSION

All the results obtained with the proposed approximations, prove the difficulty of modeling non-linear behavior of laminated beams in bending. The approximate approach adopted in this work improves the results compared to those of the literature search. Furthermore, the thickness ratio *l*/*h* is a primary factor in geometrically non-linear bending. Indeed, the laminated composite beams with a small ratio *l*/*h* exhibit a linear behavior. For this case, the first-order shear deformation theory which takes into account the transverse shearing strain and the correction of coefficients is recommended for modeling the mechanical behavior of laminates. However, a laminated composite beam with a relatively large thickness ratio *l*/*h* presents a non-linear behavior. Generally, modelization in this work following this approximate approach is strongly recommended, due to its simplicity in application and its admissible results. In particular, the average global stresses method calculates the equivalent modulus of elasticity with more precision. That explains the small difference from the experimental curve which is 1%. The results, provided by this analytical model under large deflection, remain admissible by representing an error inferior to 11%. Whereas, the apparent bending modulus method presents a large difference which is the order of 7.5% in low deflection and 20% in large deflection that is justified by the approximations adopted by this method for the calculation of the equivalent modulus of elasticity.

5. REFERENCES

Aydogdu, M., 2009. A New Shear Deformation Theory for Laminated Composite Plates. *Composite Structures*, Volume 89, pp. 94–101

Dash, P., Singh, B., 2010. Geometrically Nonlinear Bending Analysis of Laminated Composite Plate. *Commun Nonlinear Sci Numer Simulat*, Volume 15, pp. 3170–3181

Echaabi, J., Trochu, F., Pham, X., Ouelette, M., 1996. Theoretical and Experimental Investigation of Failure and Damage Progression of Quasi-isotropic Graphite Epoxy

Composites in Flexural Bending Test. *Journal of Reinforced Plastics and Composites*, Volume 15(7), pp. 740–755

Fares, M.E., Elmarghany, M.Kh., 2008. A Refined Zigzag Nonlinear First-order Shear Deformation Theory of Composite Laminated Plates. *Composite Structures*, Volume 82, pp. 71–83

Gay, D., 1997. *Matériaux Composites*. Editions Hermès, 3e édition 672 pages

Irhirane, E.H., Abousaleh, M., Echaabi, J., Hattabi, M., Saouab A., Bensalah, M.O., 2010. Modeling and Simulation of the Rupture and Rigidity Degradation of a Graphite Epoxy in Three Points Bending Test. *Journal of Engineering Materials and Technology*, Volume 132(3), pp. 031013–031018

Irhirane, E.H., 2007. *Modélisation et analyse de l'endommagement et de la résistance des matériaux composites à fibre longues*. Thèse de doctorat, École Nationale Supérieure d'Électricité et de Mécanique, Casablanca

Moreno, M., Gutiérrez A., Martínez, V., 2016. Different Response under Tension and Compression of Unidirectional Carbon Fibre Laminates in a Three-point Bending Test. *Composite Structures*, Volume 136, pp. 706–711

Nguyen-Van, H., Nguyen-Hoai, N.T., Chau-Dinh, T., Nguyen-Thoi, 2014. Geometrically Nonlinear Analysis of Composite Plates and Shells via a Quadrilateral Element with Good Coarse-mesh Accuracy. *Composite Structures*, Volume 112, pp. 327–338

Thai, H.T., Kim, S.E., 2015. A Review of Theories for the Modeling and Analysis of Functionally Graded Plates and Shells. *Composite Structures*, Volume 128, pp. 70–86

Thai, H.T., Choi, D., 2013. A Simple First-order Shear Deformation Theory for Laminated Composite Plates. *Composite Structures*, Volume 106, pp. 754–763

Tran, L.V., Lee, J., Nguyen-Van, H., Nguyen-Xuan, H., Wahab, M.A., 2015. Geometrically Nonlinear Isogeometric Analysis of Laminated Composite Plates based on Higher-order Shear Deformation Theory. *International Journal of Non-Linear Mechanics*, Volume 72, pp. 42–52

Vannucci, P., Verchery, G., 2001a. A Special Class of Uncoupled and Quasi-homogeneous Laminates. *Composite Science and Technology*, Volume 61, pp. 1465–1473

Vannucci, P., Verchery, G., 2001b. Stiffness Design of Laminates using the Polar Method. *International Journal of Solids and Structures*, Volume 38, pp. 9281–9294

Vargas, G., Mujika, F., 2011. Determination of In-plane Shear Properties by Three-point Flexure Test of 45 Anti-symmetric Laminates. *Polymer Testing*, Volume 30, pp. 204–215

Venetis, J.C., Sideridis, E.P., 2015. Approximate Solution to Three Point Bending Equation for a Simply Supported Beam. *Scientific Research and Essays*, Volume 10(9), pp. 339–347

Vong, T.S., Verchery, G., 1986. Une méthode d'aide graphique à la conception des séquences d'empilement dans les stratifies (Graphical Aided Design of Stacking Sequences for Laminates). *In*: Proceedings of the 5th French conference on composite materials (JNC-5), Paris, pp. 267–280

Werren, F., Norris, C.B., 1959. *Mechanical Properties of a Laminate Designed to be Isotropic*. Wood Engineering Research Forest Products Laboratory Madison, Wisconsin, 53705, Information Reviewed and Reaffirmed, No. 1841

Xiwen, J., Zihui, X., Bohong, G., 2013. Numerical Analyses of 3D Orthogonal Woven Composite under Three-point Bending from Multi-scale Microstructure Approach. *Computational Materials Science*, Volume 79, pp. 468–477

Zhang, Y.X., Cheung, Y.K., 2003. Geometric Nonlinear Analysis of Thin Plates by a Refined Nonlinear Non-conforming Triangular Plate Element. *Thin-Walled Structures*, Volume 41, pp. 403–418

Zhang, Y.X., Kim, K.S., 2005. A Simple Displacement-based 3-node triangular Element for Linear and Geometrically Nonlinear Analysis of Laminated Composite Plates. *Computer Methods in Applied Mechanics and Engineering*, Volume 194, pp. 4607-4632