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DEVELOPMENT OF THE DKMT ELEMENT FOR ERROR ESTIMATION IN COMPOSITE PLATE STRUCTURES

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ABSTRACT

This paper presents an application of the Discrete Kirchhoff-Mindlin Triangular (DKMT) element for error estimation in composite structures. The DKMT element passed the patch tests and gave good results in many plate bending applications. The DKMT element formulation in composite application uses the same technique as the Discrete Kirchhoff-Mindlin Quadrilateral (DKMQ) composite introduced. The benchmark tests for composite plates have been analyzed, as validation, using the methods employed by Srinivas (1973) and Pagano (1970). The DKMT plate bending element gave a good performance in convergence tests and can be used as one of tools in analyzing composite structures. Moreover, error estimation using various recovery methods such as Averaging, Projection and Superconvergent Patch Recovery (SPR) has been studied. All recovery methods used give similar results.

Keywords: Averaging; Composite; DKMT; Projection; Superconvergent Patch Recovery

1. INTRODUCTION

The development of modern technology is centered on composites as the main materials. In civil, mechanical and aerospace engineering applications, composite structures are widely used. They provide a stiffness maximum with a minimum of weight. The researches in experimental and numerical analysis of composite structures are still being conducted. Compared to other materials, composite materials are very light, yet very strong. Composite agro-material, made from renewable materials, now has been developed and can be used in large a number of construction projects. A sustainable future will be achieved by using composites as the main material in engineering construction. For this reason, a computational method is required to support the analysis of composite structures. Moreover, good precision computational methods in composite structures are crucial.

The Finite Element Method (FEM) is a numerical method used to solve various problems in structures, soil mechanics, fluids, etc. Since FEM is an approximation method where the exact solution is estimated using repetition in the discretisation process by increasing the number of elements or refining the element size, a discretisation strategy must be taken into account for each mesh refinement process. The margin of error at each mesh size can only be estimated, since no exact solution exists.

A complex problem usually has no exact solution; therefore, the error produced is also difficult to determine. An error estimator has been developed to gain a solution as close as possible to

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the exact solution.

Zienkiewicz and Zhu (1987) have introduced an error estimator, known as 'Z^2', which is simple and can be applied easily in FEM programs. Moreover, they also presented a recovery method for averaging and projection. Zienkiewicz and Zhu (1989) have applied the error estimator Z^2 in plate bending problems. Using triangular elements with uniform and adaptive meshes, they found that error estimator Z^2 is very effective. The first superconvergent method is the Superconvergent Patch Recovery (SPR) method, which was developed by Zienkiewicz and Zhu (1992a; 1992b; 1992c). The basic principle of this method is about recovering element modal forces by the least square fit. Boroomand and Zienkiewicz (1997a; 1997b). Boroomand et al. (2004) have proposed another superconvergent method called Recovery by Equilibrium in Patches (REP). This method is based on equilibrium of the solution to produce recovered internal forces. A recovery method called Recovery of Stresses by Compatibility in Patches (RCP) has been introduced (Ubertini, 2004).

DKMT and DKMQ elements which are able to analyze thick to thin plate bending problems have also been introduced (Katili, 1993a; 1993b). The formulation of DKMQ and DKMT plate elements are based on the Reissner-Mindlin hypothesis which only require C^0 continuity, (Reissner, 1943; Mindlin, 1951). DKMQ and DKMT elements are free of shear locking and their capability has been tested (Katili, 1993a; 1993b). They give a good result for the problems of thin to thick plate. Therefore, there is a great interest in applying the formulation of the DKMT element for composite structures.

In this paper, we will analyze the application of the DKMT element for composite plate structures. The same technique as used in the DKMQ element for composite applications that will be used for the DKMT element (Katili et al., 2015). The results proposed by Pagano (Pagano, 1970; Pagano & Hatfield 1972) and Srinivas, (1973) will be used to validate this formulation. And then, error estimation for composite plate structures will be analyzed using the error estimator Z^2. Averaging, projection and SPR methods will be addressed as a recovery method in this paper.

2. METHODOLOGY

2.1. Formulation of DKMT Orthotropic

The DKMT element was developed by Katili (1993a). It has three nodes and three *d.o.f* per node only. The DKMT is a very good triangular element which can take into account the transverse shear strain. It gives a good result in isotropic plate problems. It passed the patch test and gave good results in thin to thick plate problems.

In the composite laminated plate structures, the material is formed by orthotropic layers with orthotropic axes *L*-*T*-*Z* and isotropic layers in the plane *TZ* (See Figure 1). Each layer satisfies the plane stress assumption ($\sigma_z = 0$).

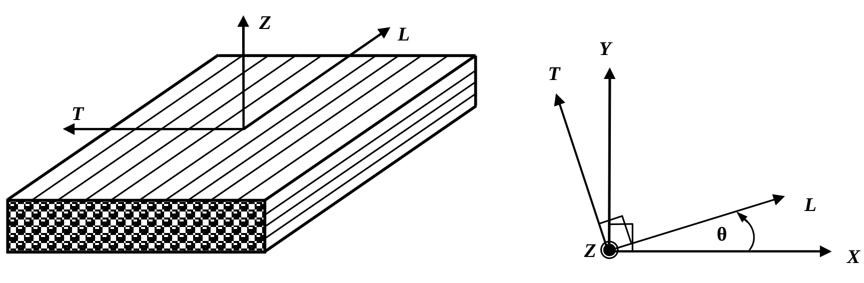


Figure 1 Layer orthotropic

In each layer, the constitutive relationships in the orthotropy axes (L-T-Z) are as shown in Equation 1:

$$\{\sigma_L\} = [H_L]\{\varepsilon_L\} \quad ; \quad \{\tau_L\} = [G_L]\{\gamma_L\} \tag{1}$$

where, as shown in Equations 2, 3 and 4:

$$\begin{aligned} \langle\sigma_L\rangle &= \langle\sigma_L \quad \sigma_T \quad \sigma_{LT}\rangle \quad ; \quad \langle\tau_L\rangle = \langle\tau_{LZ} \quad \tau_{TZ}\rangle \\ \langle\varepsilon_L\rangle &= \langle\varepsilon_L \quad \varepsilon_T \quad 2\varepsilon_{LT}\rangle \quad ; \quad \langle\gamma_L\rangle = \langle\gamma_{LZ} \quad \gamma_{TZ}\rangle \end{aligned} \tag{2}$$

$$[H_L] = \begin{bmatrix} H_{LL} & H_{LT} & 0 \\ & H_{TT} & 0 \\ sym & & G_{LT} \end{bmatrix} \quad ; \quad [G_L] = \begin{bmatrix} G_{LZ} & 0 \\ 0 & G_{TZ} \end{bmatrix} \tag{3}$$

$$\begin{gathered} H_{LL} = \frac{E_L}{1-\upsilon_{LT}\upsilon_{TL}} \quad ; \quad H_{TT} = \frac{E_T}{1-\upsilon_{LT}\upsilon_{TL}} \quad ; \quad H_{LT} = \frac{E_T\upsilon_{LT}}{1-\upsilon_{LT}\upsilon_{TL}} = \frac{E_L\upsilon_{TL}}{1-\upsilon_{LT}\upsilon_{TL}} \\ G_{LZ} = G_{LT} \quad ; \quad G_{TZ}\left(\text{or } \upsilon_{TZ} \text{ with } G_{TZ} = \frac{E_T}{2(1+\upsilon_{TZ})}\right) \end{gathered} \tag{4}$$

E_L is the Young Modulus in the fiber direction and E_T is Young Modulus in the transverse direction to the fiber. The variables υ_{LT} and υ_{TL} are Poissons' ratio in the L-T plane of orthotropy. The constitutive parameters in matrices [H_L] and [G_L] can be measured experimentally. The orthotropy direction L and T can vary for each layer and are represented by angle θ between the global axis X and the directions L_i of the i^{th} layer (Figure 1). The matrix transformation from Orthotropic to Cartesian coordinates is shown in Equation 5:

$$[R_{L1}] = \begin{bmatrix} C^2 & S^2 & CS \\ S^2 & C^2 & -CS \\ -2CS & 2CS & C^2-S^2 \end{bmatrix} ; [R_{L2}] = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} ; C = \cos\theta \; ; \; S = \sin\theta \tag{5}$$

Finally in the Cartesian coordinate systems (X-Y), we can write the Constitutive Law for i^{-th} layer as shown in Equation 6:

$$\{\sigma\} = [H]_i\{\varepsilon\} \quad ; \quad \{\tau\} = [G]_i\{\gamma\} \tag{6}$$

where stresses and strains are shown in Equation 7:

$$\begin{gathered} \{\sigma\} = \langle\sigma_x \quad \sigma_y \quad \sigma_{xy}\rangle^T ; \{\varepsilon\} = \langle\varepsilon_x \quad \varepsilon_y \quad 2\varepsilon_{xy}\rangle^T ; \; \{\tau\} = \langle\tau_x \quad \tau_y\rangle^T \; ; \; \langle\gamma\rangle = \langle\gamma_x \quad \gamma_y\rangle \\ [H]_i = [R_{L1}]^T[H_L][R_{L1}] \quad ; \quad [G]_i = [R_{L2}]^T[G_L][R_{L2}] \end{gathered} \tag{7}$$

[H_b] and [H_s] matrices homogenized using the layering technique will be given by Equations 8 and 9:

$$\left[H_b\right]=\begin{bmatrix} H_{b_{11}} & H_{b_{12}} & H_{b_{13}} \\ & H_{b_{22}} & H_{b_{23}} \\ Sym. & & H_{b_{33}} \end{bmatrix}=\frac{1}{3}\sum_{i=1}^{nl}\left[H\right]_i\left(z_{i+1}^3-z_i^3\right) \tag{8}$$

$$\left[\bar{H}_s\right]=\begin{bmatrix} \bar{H}_{s_{11}} & \bar{H}_{s_{12}} \\ Sym. & \bar{H}_{s_{22}} \end{bmatrix}=\sum_{i=1}^{nl}\left[G\right]_i\left(z_{i+1}-z_i\right) \tag{9}$$

where nl is the number of layers in the structures. $[H]_i$ is the in-plane constitutive matrix for i^{-th} layer and $[G]_i$ is the out of plane constitutive matrix for i^{-th} layer, as given by Equation 7.

Matrix $[H_s]$ is defined so that the shear strain energy density obtained for an exact 3D distribution of the transverse shear stresses τ_x and τ_y, which is identical to the shear energy associated to the Reissner-Mindlin plate model. The aims are that the transverse shear stiffness of the plate model corresponds as much as possible with that deduced from 3D analysis. We have $[H_s]$ for the constitutive equation for the shear forces in Equation 7.

$$\left[H_s\right]=\begin{bmatrix} k_{11}.\bar{H}_{s_{11}} & k_{12}.\bar{H}_{s_{12}} \\ Sym. & k_{22}.\bar{H}_{s_{22}} \end{bmatrix} \tag{10}$$

where: the variables k_{11}, k_{12} and k_{22} identify the transverse shear correction parameters, and $\left[H_b\right]$, $\left[\bar{H}_s\right]$ and $\left[H_s\right]$ are the symmetric matrices.

The method for computing the transverse shear correction parameters in Equation 10 is based on considerations of static equivalences described in Batoz & Dhatt, (1990) and Oñate, (2012). In the case of an isotropic material, the DKMT element uses a shear influence factor (ϕ_k) as shown in Equation 11 (Katili, 1993a):

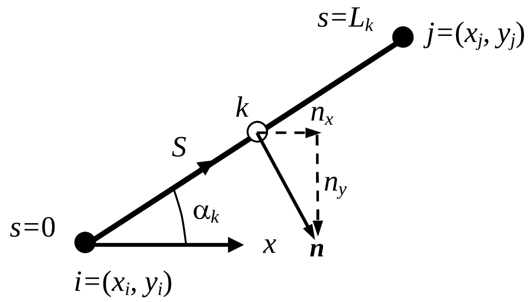
$$\phi_k=\frac{D_b}{D_s}\frac{12}{L_k^2}=\frac{2}{k\left(1-\upsilon\right)}\left(\frac{h^2}{L_k^2}\right) \tag{11}$$

The factor ϕ_k maintains the consistency of the element. The factor h^2/L_k^2, in Equation 11 explains why the DKMT element behaves as described in the *Reissner-Mindlin* theory for thick plates and as in the *Kirchhoff-Love* theory for thin plates. In the latter case, the factor h^2/L_k^2 is close to zero, so that the transversal shear deformation is automatically ignored, as a result, the *shear locking* is resolved by this method. For composite applications, we used the modified shear influence factor (ϕ_k) as follows in Equation 12:

$$\phi_k=\left(H_{sk21}^{inv}H_{bk32}+H_{sk22}^{inv}H_{bk22}\right)\left(\frac{12}{L_k^2}\right) \tag{12}$$

with : L_k as the length of the side k, $\left[H_{bk}\right]=\left[QE\right]^T\left[H_b\right]\left[QE\right]$ and $\left[H_{sk}\right]=\left[Q\right]^T\left[H_s\right]\left[Q\right]$ where $[H_b]$ and $[H_s]$ are defined in Equations 8 and 10, respectively and shown in Equation 13:

$$[QE]=\begin{bmatrix} n_x^2 & n_y^2 & -n_x n_y \\ n_y^2 & n_x^2 & n_x n_y \\ 2n_x n_y & -2n_x n_y & n_x^2-n_y^2 \end{bmatrix} \quad ; \quad [Q]=\begin{bmatrix} n_x & -n_y \\ n_y & n_x \end{bmatrix} \tag{13}$$



$C_k = \cos\alpha_k = x_{ji}/L_k$

$S_k = \sin\alpha_k = y_{ji}/L_k$

$n_x = S_k$

$n_y = -C_k$

$L_k^2 = x_{ji}^2 + y_{ji}^2$

$x_{ji} = x_j - x_i$

Figure 2 Geometry and local tangential coordinates on side *k*

We will use this new shear influence factor (ϕ_k) shown in Equation 12 for the application of the DKMT element in composite plate structure. With these methods, we are able to compute the composite structure for thick and thin plate problems. Also, it will become a tool for computation in modeling the structure with composite materials.

2.2. Recovery Methods and Error Estimator Z^2

While the FEM solution has been known to give continuity in displacement at nodal points, it yields discontinuity and inaccuracy problems when used to calculate internal forces at joined sides of the boundary elements. Yet, the nature of FEM solution, which calculates internal forces using the derivation of the displacement function, has created such a problem. This problem occurs in the finite element method that later on is the basic approach for estimating the error of finite element calculation.

Recovery methods that will be used in this paper have been detailed in Katili (2009). Since calculation will never stop if the element size is close to zero, we need an effective condition as criteria to terminate the discrete process. The factor for the relative error ϕ^* of a structure with the recovery method is shown in Equation 14:

$$\phi^* = \frac{\|e^*\|}{\|u^*\|} \times 100\% \tag{14}$$

The error indicator represents value, used as criteria to terminate the refinement process, where usually we take ϕ^*= 5 % as a limit.

3. RESULTS AND DISCUSSION

3.1. Simply Supported (SS) under Uniform Loading of a Sandwich Plate

In this test, we will analyze a simply supported, square sandwich plate (hard support conditions: $w=\beta_s=0$) under uniform loading f_z. We can see the details of this test in Figure 3. Because of symmetry, we will only analyze area *ABCD*.

Data material (skin and core orthotropic) are: E_L =3.4156 MPa; E_T =1.7931 MPa; υ_{LT} = 0.44; G_{LT} =1 MPa; G_{LZ} =0.608 MPa; G_{TZ}=1.015 MPa; Stratification: sandwich; 3 layers 0/0/0 symmetrical. The skin (Layers 1 and 3 in Figure 3) and the core are formed by orthotropic material with the same axes for the orthotropic. While the Layer 2 (core) properties are proportional to those of Layers 1 and 3, the E and G values of the core are $\boldsymbol{C}$ times weaker than those of the skin. In this test we will use $\boldsymbol{C}$ = 1, 10, and 50, respectively. For $\boldsymbol{C}$ = 1, the structure corresponds to the orthotropic plate, while for $\boldsymbol{C}$ = 10 and $\boldsymbol{C}$ = 50, it corresponds to the sandwich structure. The results of central deflection at point C are reported in Table 1. Srinivas (1973), reported on an analytical solution using 3D elasticity theory. The central displacement is expressed in the form of Equation 15:

$$\underline{w}_C = \frac{w_C\, G_{LT}\left(\text{Core}\right)}{h\, f_z} \tag{15}$$

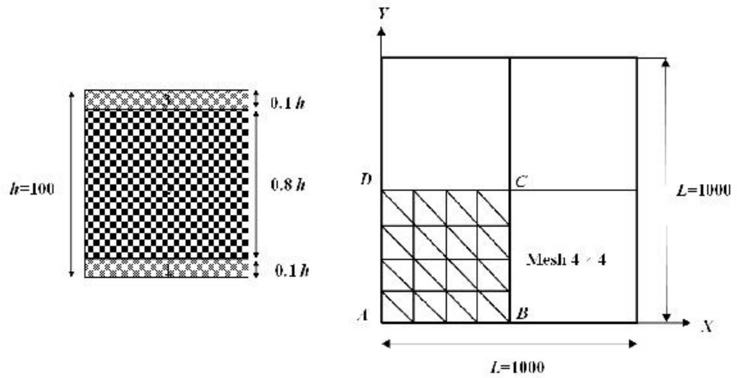


Figure 3 Simply supported square sandwich plate under uniform loading

Table 1 presents the central deflection of $\underline{w}_C$ for $\boldsymbol{C}$ = 1, 10 and 50 by using a uniform mesh. We use N×N = 2×2, 4×4, 8×8, 16×16, 32×32, 64×64 and 96×96. We found accordance in the results and they are very close to the reference solution. For the mesh N×N= 96×96, we found an error of 0.15% for $\boldsymbol{C}$ = 1; 0.2% for $\boldsymbol{C}$ = 10 and 0.5% for $\boldsymbol{C}$ = 50, respectively. It shows that a good behaviorial convergence is given by the DKMT element in a composite structure.

Table 1 Convergence of central deflection $\underline{w}_C$ (uniform mesh)

N×N	***C* = 1**	***C* = 10**	***C* = 50**
	k_{11}=k_{22}=0.8333;k_{12}=0	k_{11}=k_{22}=0.3521;k_{12}=0	k_{11}=k_{22}=0.0938;k_{12}=0
2×2	161.920	36.257	14.172
4×4	175.200	39.886	15.962
8×8	178.950	41.276	16.579
16×16	180.500	41.790	16.767
32×32	181.110	41.943	16.819
64×64	181.290	41.984	16.833
96×96	181.330	41.992	16.835
Srinivas (1973)	**181.050**	**41.910**	**16.750**

Figure 4 shows the distorted mesh used in this study as the mesh N×N = 4×4. The results for distorted mesh are presented in Table 2. We can see that the results for the mesh N×N = 96×96 are similar with that given by the uniform mesh. Moreover, this test reveals that DKMT element in a composite application is not sensitive to distortion.

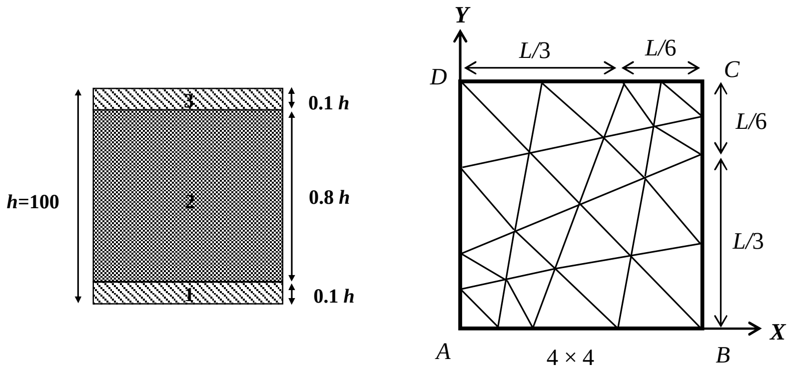


Figure 4 Distorted mesh (mesh 4×4)

3.2. A Simply Supported 9-layer Square Plate under a Sinusoidal Pressure Load

In this test, we will analyze a simply supported 9-layer unidirectional square laminated with L=1000 and the thickness varies (h = 250, 100, 20 and 10). We will use DKMT plate element with 6×6 and 10×10 mesh size. This test was proposed by Pagano and Hatfield (1972).

The sum of thickness of 0° layer ($h_1+h_3+h_5+h_7+h_9$) is equal to the sum of thickness of 90° layer ($h_2+h_4+h_6+h_8$) whereas a layer in the same direction has similar thicknesses ($h_1=h_3=h_5=h_7=h_9$) and ($h_2=h_4=h_6=h_8$), (See Figure 4).

Material properties: E_L = 25 MPa ; E_T = 1 MPa ; υ_{LT} = 0.25 ; G_{LT} = 0.5 MPa ; G_{TZ} = 0.2 MPa ; k_{11} = 0.670 ; k_{22} = 0.666 ; k_{12} = k_{21} = 0 ; Stratification : 9 layers 0/90/0/90/0/90/0/90/0 symmetrical ; $f_z = f_0 \sin(\pi x/L)\sin(\pi y/L)$.

The solution can be expressed with the following form in Equation 16:

$$\underline{w} = \frac{\text{®}^4 w Q}{12 S^4 h\, f_0} \quad ; \quad Q = 4G_{LT} + \left[E_L + E_T\left(1+2\upsilon_{TT}\right)\right] / \left(1-2\upsilon_{LT}\upsilon_{TL}\right) \quad ; \quad S = \frac{h}{L} \tag{16}$$

We can see in Table 3 the central displacement and stresses at point C compared with the solution given by Pagano and Hatfield (1972). We found a good correlation between the DKMT element and the reference solution by using a very few elements.

Table 2 Convergence of central deflection $\underline{w}_C$ (distorted mesh)

N*×*N	***C* = 1**	***C* = 10**	***C* = 50**
	$k_{11}=k_{22}$=0.8333;k_{12}=0	$k_{11}=k_{22}$=0.3521;k_{12}=0	$k_{11}=k_{22}$=0.0938;k_{12}=0
2×2	168.170	38.123	15.266
4×4	176.920	40.512	16.317
8×8	179.570	41.488	16.685
16×16	180.710	41.851	16.796
32×32	181.170	41.959	16.827
64×64	181.310	41.988	16.835
96×96	181.330	41.993	16.836
Srinivas (1973)	**181.050**	**41.910**	**16.750**

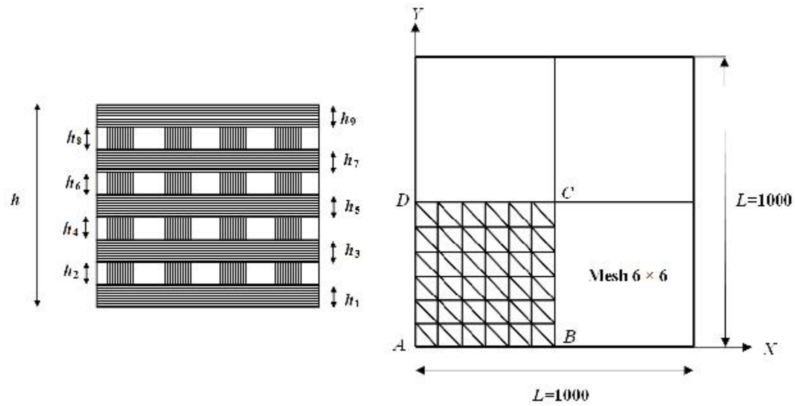


Figure 5 Simply supported 9-layer (0/90/0/90/0/90/0/90/0) square plate under sinusoidal pressure load

Table 3 Convergence of the deflection at point *C* for 9-layer (0/90/0/90/0/90/0/90/0) square plate

	L/h	4	10	50	100
$\underline{w}$	Mesh 6×6	3.906	1.423	0.962	0.948
	Mesh 10×10	4.028	1.464	0.988	0.973
Pagano and Hatfield (1972)		**4.079**	**1.512**	**1.021**	**1.005**

3.3. Error Estimation Simply Supported (SS) of a Sandwich Plate

This test has been analyzed in (3.1); in this test we will analyze error estimation using various recovery methods and the error estimator Z^2. We use the relative error factor $\phi^* = 5$ % as a limit to terminate the refinement process. Mesh *N*×*N* = 4×4, 8×8, 16×16 and 32×32 are employed.

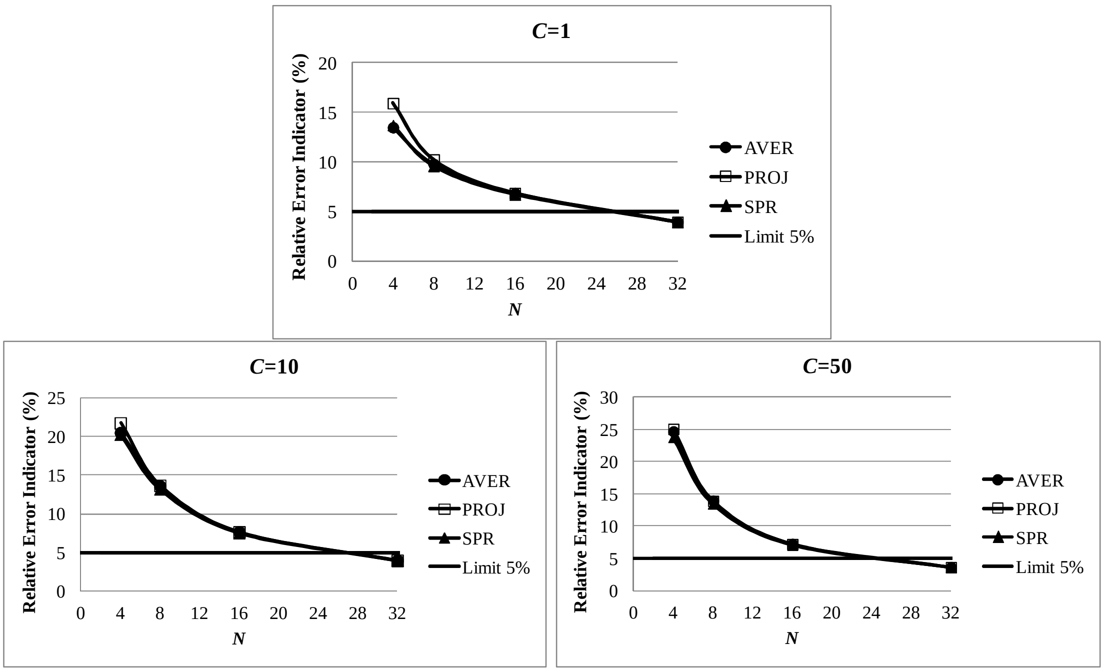


Figure 6 Relative Error Indicator ϕ^*

For the mesh 32×32, we can see in Figure 6 the results of relative error ϕ^* which passed the limit of 5%. All recovery methods give the similar results for the problem analyzed. Table 4 presents the result of $\underline{\sigma}_{XC}$, as expressed in Equation 17 for various recovery methods. For ***C***=1, we found Averaging (AVER) method gives the results superior to the SPR and Projection (PROJ) methods. However, for ***C***=10 and ***C***=50, the recovery methods used give similar results.

$$\underline{\sigma}_{XC} = \frac{\sigma_{XC}}{f_z} \tag{17}$$

Table 4 $\underline{\sigma}_{XC}$ from different recovery methods

C	**Method**	**$\underline{\sigma}_{XC}$(-2h$^+$/5)**	**Error(%)**	**$\underline{\sigma}_{XC}$(-2h$^-$/5)**	**Error(%)**	**$\underline{\sigma}_{XC}$(-h/2)**	**Error(%)**
1	AVER	28.545	0.001	28.545	0.001	35.681	0.721
	PROJ	28.539	0.020	28.539	0.020	35.674	0.740
	SPR	28.594	0.170	28.594	0.170	35.742	0.551
	Srinivas (1973)	**28.545**		**28.545**		**35.940**	
10	AVER	5.062	4.152	50.618	4.131	63.273	2.777
	PROJ	5.061	4.134	50.609	4.113	63.261	2.794
	SPR	5.072	4.354	50.716	4.332	63.395	2.589
	Srinivas (1973)	**4.860**		**48.610**		**65.080**	
50	AVER	25.795	46.544	25.288	58.180	13.034	25.795
	PROJ	25.762	46.532	25.254	58.165	13.057	25.762
	SPR	26.038	46.634	25.529	58.292	12.866	26.038
	Srinivas (1973)	**0.740**		**37.150**		**66.900**	

4. CONCLUSION

The application of the DKMT element for error estimation in composite structures has been presented. We found results which are very close to the reference solution (the relative error is under 5% for a few elements) for all proposed tests to validate the composite structure. The DKMT element affords good convergence behavior and it is not sensitive to distortion. Recovery methods used in this paper give similar results and are close to the reference solution for ***C***=1 and ***C***=10 (with a relative error under 5%). However, for ***C***=50 we found an important error. Finally, the DKMT plate bending element can be used as a tool to analyze composite structures.

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